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LAMINAR FILM FLOW DOWN A WAVY INCLINE

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Abstract—Laminar flow of a liquid, down an inclined wall with sinusoidal corrugations, is considered. A linear analysis, valid for small-amplitude disturbances but arbitrary wavelength and Re number, leads to an Orr–Sommerfeld type equation with nonhomogeneous boundary conditions. The free-surface amplitude and phase relative to the wall are examined. In a range of Re numbers, a resonance phenomenon is calculated, leading to amplification of the wall corrugations. This behavior has not been encountered in previous analyses of thin film flow, based on the Stokes approximation. Copyright © 1996 Elsevier Science Ltd.

Key Words: film flow, wall corrugations, free surface resonance

1. INTRODUCTION

The flow of a liquid film down an inclined wall is a topic with intriguing theoretical ramifications and many useful applications. Flow of this kind occurs in heat-transfer equipment, such as falling film evaporators and condensers. It is also of interest with relation to mass-transfer in absorption columns using packing, in particular under conditions away from flooding, where the shear imposed by the countercurrent gas flow can be neglected (see for example the liquid side, mass-transfer correlations proposed by Fair and Bravo 1990).

Fundamental studies of this flow date back to the classical work of Kapitsa and Kapitsa (1949) and concentrate on the linear stability of the flat film and the nonlinear wave dynamics. Early work in this area is summarized in a review article by Fulford (1964) and an exposition of recent results, based on long-wave expansions of the Navier–Stokes equations, is provided by Chang (1994).

A variation of this problem involves film flow down a wavy—rather than a flat—wall. The main practical interest stems from efforts to enhance interfacial transfer rates through modifications of the basic flow field imposed by the wall undulations. Structured packing (consisting of corrugated metal sheets) offers a typical example.

Studies of flow down inclined, periodic walls are comparatively limited. Wang (1981) performed an asymptotic analysis for sinusoidal wall with small-amplitude variations. Dassori *et al.* (1984) extended the analysis to the separated flow of two fluids through a sinusoidal channel. Pozrikidis (1988), using a boundary-integral method appropriate for creeping flow, numerically computed results for wall variations of arbitrary amplitude and shape.

Shetty and Cerro (1993) proved (by asymptotic analysis valid in the limit of negligible inertia and capillary effects) that the flow of a viscous liquid down a wavy surface obeys a local Nusselt solution with continuously varying inclination when the film thickness is much smaller than the amplitude and wavelength of the solid waves. Agreement with their measurements, and the measurements taken by Zhao and Cerro (1992), was more or less satisfactory depending on the validity of the above assumptions.

All the aforementioned works are applicable to flow with Re number equal to zero, thus neglecting inertial effects. They share the common result that the free surface profile has a phase shift relative to the wall and an amplitude that is always smaller or equal to that of the wall undulations.

The complementary problem of inertia dominated flow over a wavy bottom has been tackled in terms of inviscid theory. Horizontal, uniform base flow was considered, and the results were used mainly as an input for computations of sediment transport in rivers and Bragg scattering of surface waves (Kennedy 1963; Mei 1969; Miles 1986; Bontozoglou *et al.* 1991; Sammarco *et al.* 1994).

Linear inviscid theory predicts that for a liquid velocity, U, equal to

$$U = (g/k)\tanh kh, \tag{1}$$

resonance takes place between the stationary surface wave and the bottom forcing, leading to free-surface amplitude considerably larger than that of the wall. Terms h and k in [1] are the film thickness and wavenumber of the wall corrugations respectively. It is not presently known—and is of evident importance—whether this behavior has a counterpart in the viscous flow of thin films and whether it manifests itself in flows with inclination relative to the horizontal.

In the present work, it is attempted to answer these questions by considering liquid flow down a sinusoidal wall of arbitrary orientation. The wall amplitude, a, is assumed to be very small compared to the thickness of the liquid film. However, no assumption is made about the wavelength of the corrugations and the Re number of the flow, apart from the fact that it is taken to be laminar.

Flow variables are expanded with respect to the small parameter $\epsilon = a/h$ and the zero and first order problems are considered. The latter is solved numerically by a finite difference method.

2. FORMULATION OF THE PROBLEM

Two-dimensional flow is considered, down a wall with an average inclination φ relative to the vertical direction (figure 1). The wall is covered with small-amplitude sinusoidal corrugations at right angles to the flow direction. The flow is described by an x, y coordinate system, with y normal to the mean flow direction. The beginning of the y-axis is set at the mean wall level and the corrugations are described by the equation

$$w(x) = a\cos kx. \tag{2}$$

The mean film thickness is equal to h and the free surface of the liquid is located at

$$\eta(x) = h + f(x). \tag{3}$$

The stationary free surface shape, $f(x)$, is to first order described by the linear relation

$$f(x) = \beta a\, e^{ikx} \tag{4}$$

where β is the amplification of the wall corrugations and is in general complex. This expression corresponds to waves of the same wavelength as the disturbances but of arbitrary amplitude and phase relative to the wall.

Unit vectors $\mathbf{n}$ and $\mathbf{t}$, locally normal and tangential to the free surface, are defined in terms of the free surface slope, $f'(x)$, by the expressions

$$\mathbf{t} = (1, f')/\sqrt{1 + f'^2} \tag{5}$$

$$\mathbf{n} = (f', -1)/\sqrt{1 + f'^2}. \tag{6}$$

The flow is described by the continuity equation and the two components of the Navier–Stokes equation. Boundary conditions are no-slip on the wall,

$$u = v = 0 \text{ on } y = w(x). \tag{7}$$

Also, the velocity normal to the stationary free surface is zero

$$\mathbf{u}\cdot\mathbf{n} = 0 \text{ on } y = \eta(x) \tag{8}$$

where $\mathbf{u} = (u, v)$. Shear and normal stresses balance to zero on the free surface giving the equations

$$(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{t} = 0 \tag{9}$$

$$(\boldsymbol{\sigma}\cdot\mathbf{n})\cdot\mathbf{n} = -p_0 + s\frac{f''}{(1 + f'^2)^{3/2}}. \tag{10}$$

Term p_0 is the ambient pressure, s if the surface tension and $\boldsymbol{\sigma}$ is the stress tensor, defined in terms of the rate of strain tensor, $\mathbf{e}$, and Kronecker $\boldsymbol{\delta}$ as:

$$\boldsymbol{\sigma} = -p\boldsymbol{\delta} + 2\mu\mathbf{e}. \qquad [11]$$

The perturbation expansion performed is based on the small variable, ϵ, defined as

$$\epsilon = a/h. \qquad [12]$$

Thus, it is assumed that the amplitude of the wall corrugations is much smaller than the film thickness. A stream function, Ψ, is defined and expanded in a perturbation series in the small variable ϵ:

$$\Psi = \Psi^{(0)} + \epsilon\Psi^{(1)} + \cdots. \qquad [13]$$

The zero-order problem corresponds to laminar flow over a flat surface, with the familiar Nusselt solution

$$\Psi^{(0)}(y) = \frac{\rho g_x}{\mu}\left(\frac{hy^2}{2} - \frac{y^3}{6}\right) \qquad [14]$$

$$u^{(0)}(y) = \frac{\rho g_x}{\mu}\left(hy - \frac{y^2}{2}\right) \qquad [15]$$

where $g_x = g\cos\varphi$ is the component of gravity along the x-axis. Following the linear response of [4], the first order stream-function term is expressed as

$$\Psi^{(1)}(x, y) = \psi(y)\,\mathrm{e}^{ikx}. \qquad [16]$$

Equation [16] is substituted in the two components of the Navier–Stokes equation, which are subsequently combined to eliminate the pressure and lead to an ordinary differential equation. The variables are nondimensionalized by using the mean film thickness, h, as the characteristic length, the volume flow rate per unit span, q, as the characteristic value for the stream-function and q/h

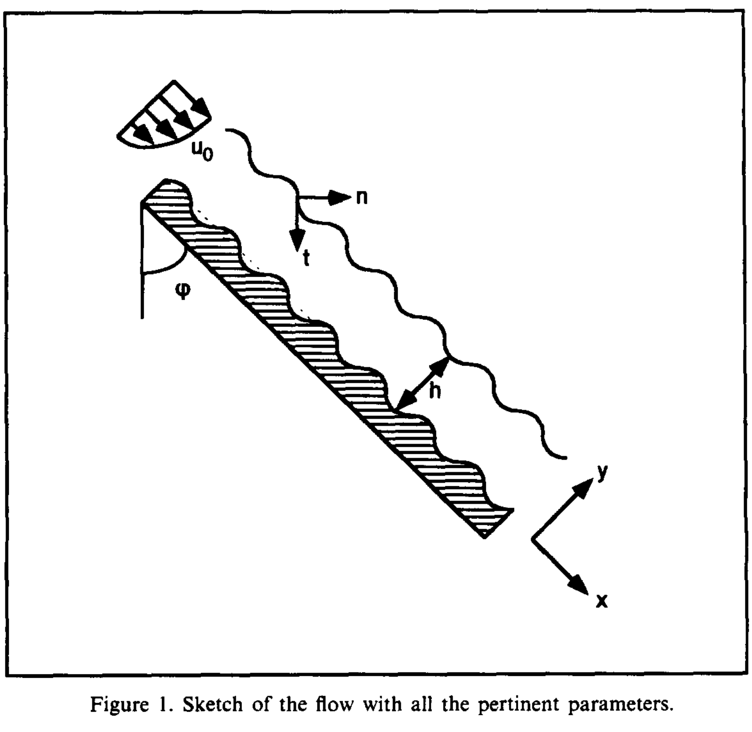

Figure 1. Sketch of the flow with all the pertinent parameters.

as the characteristic velocity. The resulting dimensionless equation is

$$\left(\frac{1}{ik\mathrm{Re}}\right)(D^2 - k^2)^2\psi = u_0(D^2 - k^2)\psi - u_0''\psi \qquad [17]$$

where Re is the Reynolds number, defined as

$$\mathrm{Re} = \frac{\rho q}{\mu} = \frac{\rho^2 g_x h^3}{3\mu^2} \qquad [18]$$

and D is the differential operator d/dy. For conciseness, the same symbols are retained for the now dimensionless variables y, D, k, ψ and u_0.

Equation [17] is the familiar Orr–Sommerfeld equation of linear stability without the phase velocity parameter. However, the present problem is not an eigenvalue problem, since not all boundary conditions are homogeneous.

According to a well-known technique, the boundary conditions along the wall and the free surface are expanded in Taylor series around the respective mean levels ($y = 0$ and $y = 1$). The final expressions are:

$$\psi(0) = 0 \qquad [19a]$$

$$\psi'(0) = -3 \qquad [19b]$$

$$\psi(1) = -3\beta/2 \qquad [19c]$$

$$\psi''(1) - k^2\psi(1) = 3\beta \qquad [19d]$$

$$\tfrac{1}{2}\mathrm{Re}\psi'(1) + \frac{i}{3k}\psi'''(1) - ik\psi'(1) = -\beta\left(\frac{sk^2}{\rho g_x} + \frac{g_y}{g_x}\right). \qquad [19e]$$

The two dimensionless groups that were used by Wang (1981) in the description of creeping flow are—in the present notation—the dimensionless wavenumber, $k = 2\pi h/L$, and the term

$$S = \left(\frac{sk^2}{\rho g_x} + \frac{g_y}{g_x}\right). \qquad [20]$$

In the present approach, these groups are supplemented by the Reynolds number of the flow, which appears in the differential equation and the boundary condition [19e].

3. NUMERICAL SOLUTION

Equation [17] with the boundary conditions [19a]–[19e] is discretized, by N equally spaced points from wall to free surface, and is solved by a centered, finite-difference scheme. At each point, the value of the unknown stream function amplitude, $\psi(y)$, is split into a real and an imaginary part. Four additional (ficticious) points—two on each side of the y region—are added to discretize the third and fourth derivatives. The last two unknowns are the real and imaginary part of the free surface amplitude, β. Thus, the total number of unknowns is $2(N + 5)$. Equation [17] is applied to the N points across the liquid film and the boundary conditions provide another five equations. Setting the real and imaginary parts to zero leads to 2 $(N + 5)$ linear algebraic equations in the unknowns.

The value of $N = 21$ was used for the derivation of the numerical results presented. Test runs performed with finer discretization ($N = 41$, 61) indicated that the solution with 21 nodes is accurate enough and the additional computational load is not justified for the purposes of the present work. This is demonstrated in figure 2, where representative points—computed with $N = 41$—coincide with the rest of the results.

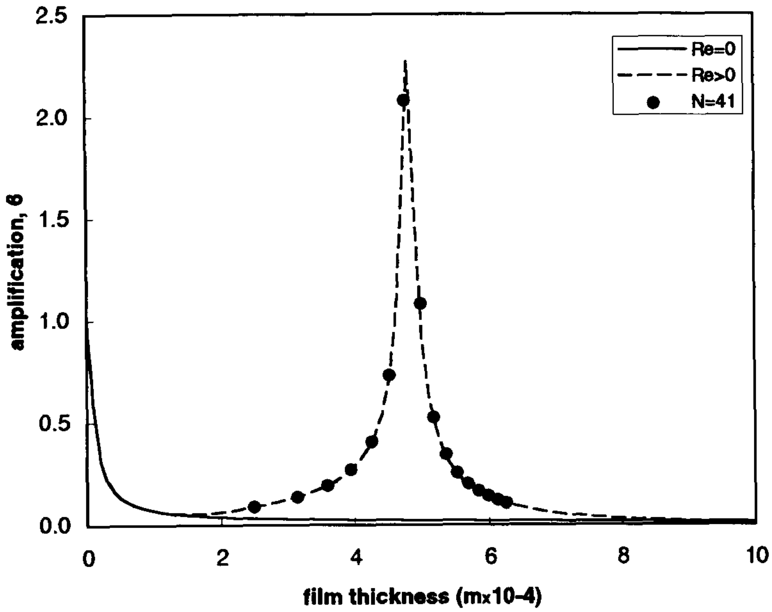

Figure 2. Comparison of the free surface amplification predicted by laminar and by creeping-flow theory for $L = 0.002$ m and $\varphi = 60°$. Circles correspond to results using a discretization of $N = 41$ points across the film.

4. RESULTS

Flow of water at temperature 20°C is used as the case study. The range of Re numbers investigated extends from 0 to 400. This choice exausts the region where a laminar solution is expected to offer realistic estimates. It may actually be argued that, at the higher Re numbers considered, the base flow will be wavy because of the growth of linearly unstable disturbances independent of the corrugated wall. However, in the linear approximation the two phenomena are separable. Therefore, the present analysis is still valid and the final flow is the result of linear superposition.

Different inclinations of the corrugated wall were examined. It turns out that the outcome is qualitatively similar for all but the almost horizontal case. Therefore, detailed results are presented only for the representative case of a wall at an angle $\varphi = 60°$ with the vertical. The parameters examined are the free surface amplification, β, and the phase shift, γ, relative to the wall corrugations.

In the limit of $\mathrm{Re} = 0$, the results of Wang (1981) are recovered. However, for nonzero Re, a totally different behavior is computed. In particular, a region of Re numbers exists, for which the interfacial amplitude is significantly higher than that of the wall corrugations. This phenomenon corresponds to an amplification of the wall structure by the flow and does not appear in computations based on the Stokes flow assumption.

A comparison of representative results calculated by Stokes flow and by the present approach appears in figure 2. The independent variable is the liquid film thickness (proportional to $q^{1/3}$ or $\mathrm{Re}^{1/3}$), and the specific case corresponds to a wall covered with sinusoidal waves with wavelength equal to 0.002 m.

In the Stokes limit, the free surface amplification, β, drops asymptotically to zero with increasing liquid film thickness. This is a universal characteristic of creeping flow, verified by the asymptotic analysis of Wang (1981) for infinitesimal waves and by the numerical computations of Pozrikidis (1988) for large-amplitude corrugations. The characteristic dimensionless quantity of this asymptotic change is the ratio of liquid film height to wavelength of corrugations. For example, for h/L of order one, the free surface amplitude is practically equal to zero. It is noted that the Re number depends on h, but not on L. Therefore, when considering a wall with very short waves the Stokes behavior is compressed to the beginning of the y-axis and the free surface is flat for small flow rates.

The results for laminar flow are computed by using, in [17] and [19a]–[19e], the value of the Re number corresponding to base flow with film thickness, h. As seen in figure 2, the computations with Re = 0 and Re > 0 coincide in the limit $h - > 0$. The rapid approach of the free surface amplitude to zero is indicative of the limited significance of wall undulations in Stokes flow.

The above picture changes drastically for thicker liquid films (higher Re number), where a significant amplification ($\beta > 1$) appears in the calculation with nonzero inertia. This behavior is reminiscent of the resonance of horizontal inviscid flow, described in the introduction. It is noted that, in the inviscid case, crossing of the resonance conditions is accompanied by an 180° change of the wave phase. More specifically, the interface is 180° out of phase at small liquid velocities and then shifts in phase with the wall corrugations (Kennedy 1963). As will be shown next, a similar discontinuity in phase shift is calculated in the present case.

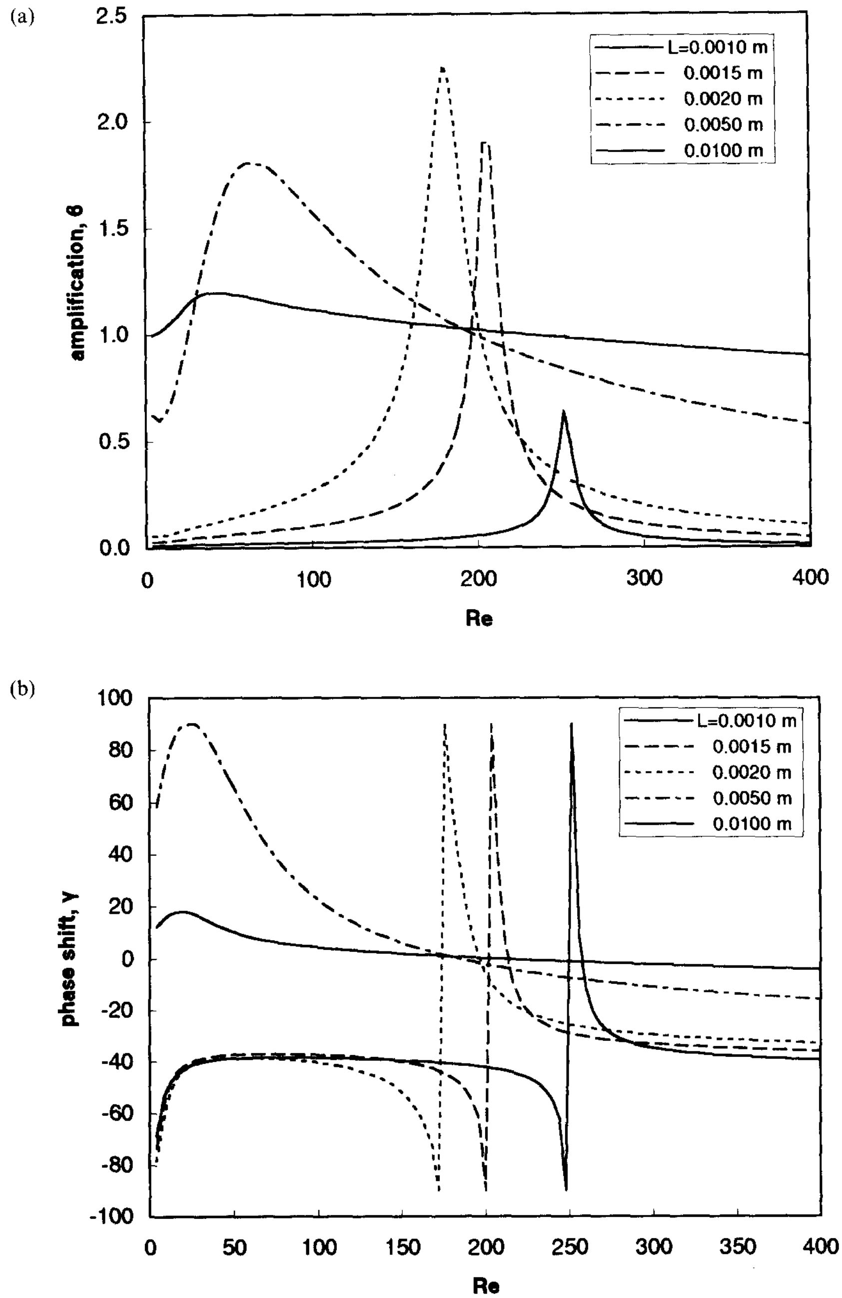

Figure 3. (a) The free surface amplification as a function of Re number, for five different wall wavelengths. (b) The free surface phase-shift as a function of Re number, for five different wall wavelengths.

A systematic investigation of the effect of wall wavelength on the amplitude and phase-shift of the free surface is presented (always for a wall with $\varphi = 60°$ to the vertical) in figure 3(a),(b). Results for five representative wavelengths, covering one order of magnitude, are plotted. Maximum amplification appears for a wavelength roughly equal to 0.002 m. For shorter waves, the resonant-like behavior is retained with gradually decreasing peak. On the contrary, for longer waves, the Re range with high amplitudes becomes wider and the sharpness of the interaction rapidly diminishes.

The phase of the free surface relative to the wavy wall is depicted in figure 3(b), and has a far more complicated behavior in the laminar than in the zero-Re flow. For the more interesting case of near-resonance, the free surface lags behind the wall ($\gamma < 0$) for small Re, reaches a maximum value and then sets up to return to zero. In the limit of Stokes flow, previous investigators noted, indeed, an asymptotic approach to $\gamma = 0$ with increasing film thickness. However, for Re > 0, figure 3(b) indicates that γ levels off to a value around $-40°$ irrespective of the wavelength of the corrugations. As the resonance conditions are approached, γ drops to $-90°$ and jumps in a dicontinous way to $+90°$. Finally, it rapidly levels off to a value between $-30°$ and $-40°$, again roughly independent of wavelength.

An inspection of figure 3(a) and (b) shows that the discontinuous jump in the value of γ occurs at the same Re number which witnesses maximum amplification. The absolute value of the jump is again 180°, though the phase before and after the transition does not remain constant. The picture which emerges is that, with increasing Re number, a resonance interaction—similar to the classical behavior of inviscid, uniform, horizontal flow—is recovered.

An interesting difference in scale between the two phenomena must be noted; the inviscid solution is of practical significance for liquid depths at least of the order of fraction of a meter. Therefore, the stationary free surface disturbances can be accurately described as gravity waves. In the present case, the wavelengths of interest are 2–3 orders of magnitude smaller and the film thickness is even less. As a result, surface tension—rather than gravity—is the relevant restoring force. This is evident from the observed insensitivity of the results to the slope of the wall.

Longer wall waves create weak free surface maxima at smaller Re. In such cases (results for $L = 0.005$ and 0.010 m in figure 3 are typical), the surface phase anticipates the wall ($\gamma > 0$) and the discontinuous jump (characteristic of resonance behavior) does not occur. However, even for these relatively long disturbances, there is a marked deviation between laminar and creeping solution. This result is borne out by a comparison for $L = 0.01$ m, shown in figure 4(a) and (b). Indeed, the two solutions coincide in amplitude and phase up to a film thickness roughly equal to 0.2 mm and then suddenly deviate at the onset of the weak resonance.

As previously noted, the near-horizontal inclination is the only one for which somewhat different results are calculated. To indicate this, the maxima of free-surface amplifications (i.e. the coordinates of the peaks in figure 3(a)) are plotted in figure 5 as a function of corrugation wavelength, for a wall with inclination 0, 60 and 89.43° relative to the vertical direction.

The maximum amplitudes, appearing in figure 5, practically coincide in the first two cases of a vertical and of an inclined wall. The overall maximum in the free-surface amplitude appears at the same wavelength. Its value abruptly drops to zero for shorter waves and asymptotically approaches the value one for waves longer than the resonant. This asymptotic behavior, combined with the observation that—for longer waves—the phase-shift is practically zero (see figure 3(b)), leads to the conclusion that, in these cases, the free surface is a replica of the wall morphology. As with all the results of the present work, this conclusion is valid in the limit of wall amplitude small relative to the film thickness.

For the near-horizontal wall, it is observed (again from figure 5) that maximum amplification occurs for longer waves ($L = 0.008$ m). These waves are expected to be influenced to a small extent by the restoring force of gravity at right angles to the flow direction. The maximum free-surface amplitude again drops to zero for shorter corrugations, but, in the opposite limit of longer waves, there is a persistant amplification extending deep into the capillary-gravity wave regime.

It is noted that points in figure 5 have different Re numbers. The Re number corresponding to maximum amplification of a specific wall wave is shown, for the three different inclinations, in figure 6. The resonance Re number is seen to decrease abruptly—from a large value for very short

(a)

(b)

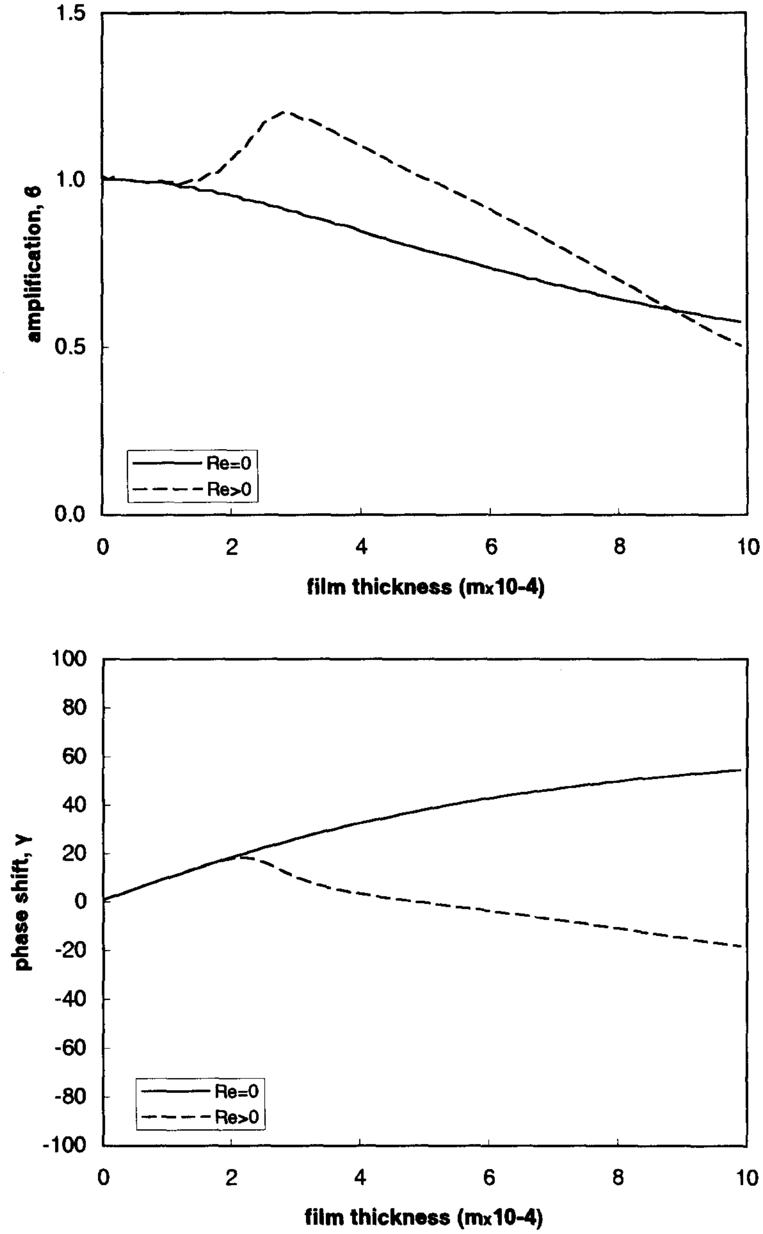

Figure 4. (a) Comparison of the free surface amplification predicted by laminar and by creeping-flow theory for $L = 0.010$ m and $\varphi = 60°$. (b) Comparison of the free surface phase-shift predicted by laminar and by creeping-flow theory for $L = 0.010$ m and $\varphi = 60°$.

corrugations—and approach an almost constant smaller value for longer waves. Again the behavior is the same for all but the near-horizontal inclination.

A mechanism responsible for the above described phenomenon is sought, in terms of resonant forcing. Following the inviscid theory of uniform, horizontal flow, it is suggested that energy is fed from the wavy wall to the free-surface structures when the flow velocity of the film agrees with the natural phase velocity of the waves. To put this suggestion in quantitative form, the mean liquid velocity, q/h, is chosen as a representative film velocity. The liquid flow rate, q, used ($q = \mathrm{Re}\cdot v$) is the value corresponding to resonance. The phase speed of free surface waves—as described by the equation

$$c = \left(\frac{g}{k}\sin\varphi + \frac{\sigma k}{\rho}\right)\tanh(kh) \qquad [21]$$

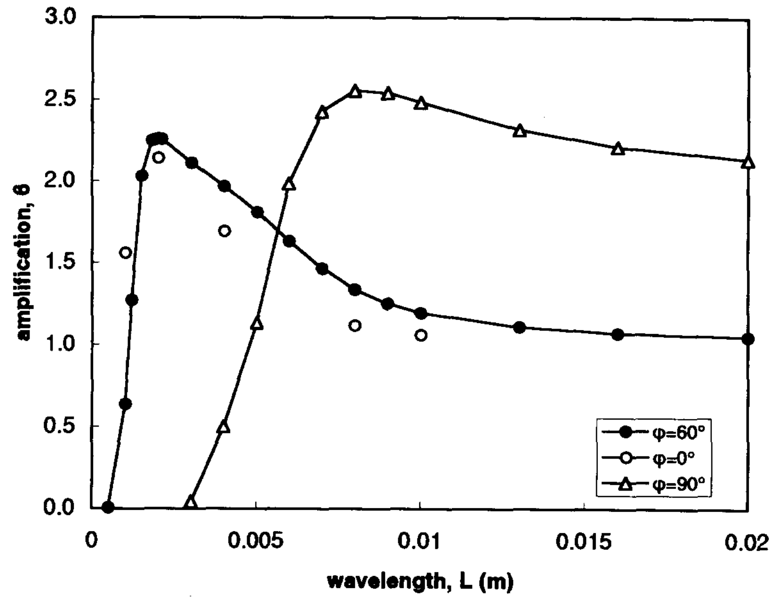

Figure 5. The maximum free surface amplification as a function of wall wavelength.

of inviscid water wave theory—is compared with the mean film velocity in figure 7. It is readily observed that the two velocities follow a similar behavior, and are very close in the region of wavelengths where strong resonance is calculated. This agreement provides evidence in support of the above tentative mechanism. It is stressed though that (unlike the inviscid case) the liquid flow may not safely be characterized by a single velocity.

5. CONCLUDING REMARKS

Laminar flow of a liquid down an inclined corrugated wall is considered. An analysis is performed, valid for small-amplitude disturbances but arbitrary wavelength and Re number. An amplification of the wall corrugation at the free surface is calculated under certain conditions. The

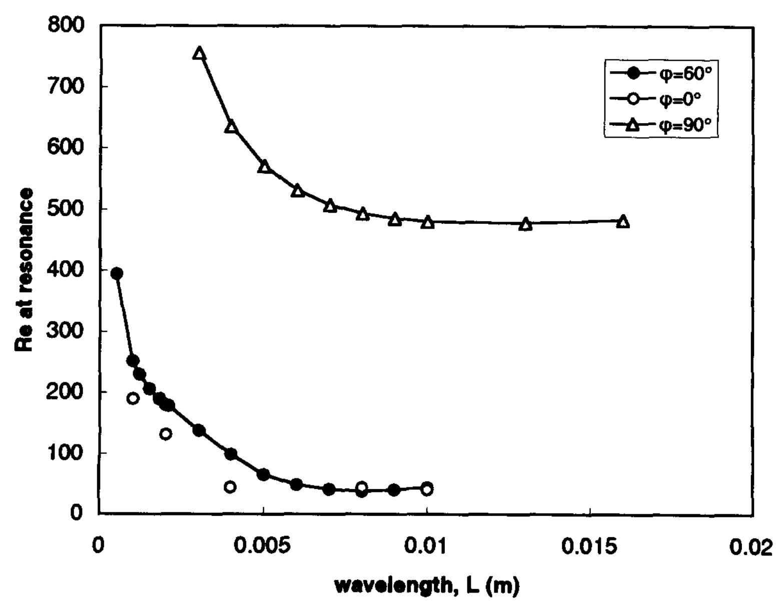

Figure 6. The Re number at maximum amplification as a function of wall wavelength.

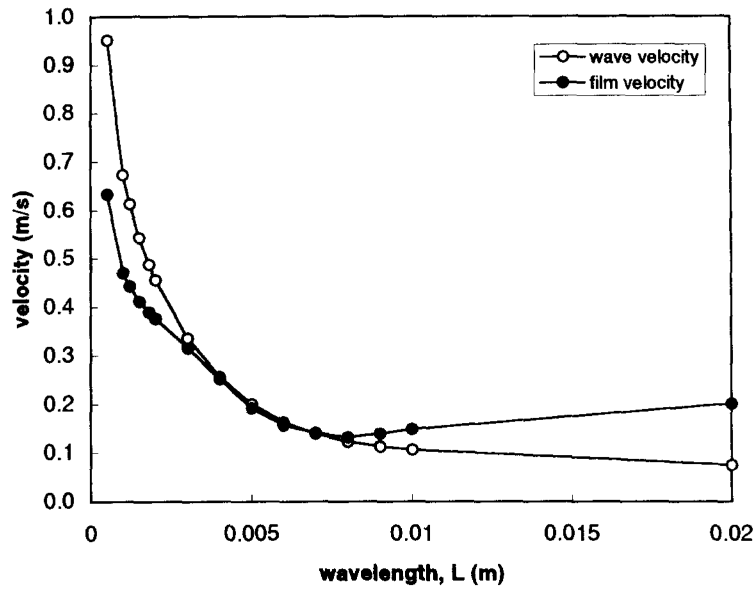

Figure 7. Comparison between the mean film velocity and the phase velocity of free surface waves with the wall wavelength.

variation of amplification ratio with the liquid flow rate points to a resonance phenomenon and a tentative mechanism is suggested.

For vertical and inclined walls, wavelengths in the range of 0.001–0.005 m are the most active in producing strong interaction with the free surface. The picture which emerges is that shorter corrugations are smoothed out be the flow, whereas much longer corrugations are simply doublicated by the free surface shape. In the resonant cases, a discontinuous change in the phase relative to the wall is observed. This behavior could be associated with small-scale variations of the wall shear stress and of the liquid velocity at the free surface, with interesting implications in heat and mass transfer processes. For a near-horizontal wall, a qualitatively similar behavior is calculated, transposed to slightly longer waves.

The small characteristic length of the described interaction points to the importance of the wall microstructure. In this context, it is interesting to note that the small-scale geometry of structured packings (used in mass-transfer equipment) and of heat-transfer surfaces has been shown by practice to have a significant effect on their performance.

The present results are of theoretical interest as they indicate a striking qualitative difference between Stokes flow and low Re laminar flow. Their practical significance depends on the nonlinear behavior of the system. Indeed, the amplifications computed are finite and wall amplitudes satisfying the linearity constraint ($a/h \ll 1$) are too small to make the linear phenomenon observable. However, if similar amplifications prevail for higher (nonlinear) wall disturbances, it is reasonable to expect that the phenomenon will be easily observed. This is an open question which provides motivation both for nonlinear analysis and experiments.

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